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Pauli equation and the method of supersymmetric factorization

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Abstract

We consider different variants of factorization of a 2×2 matrix Schrödinger/Pauli operator in two spatial dimensions. They allow us to relate its spectrum to the sum of spectra of two scalar Schrödinger operators, in a manner similar to one-dimensional Darboux transformations. We consider both the case when such factorization is reduced to the ordinary two-dimensional supersymmetric quantum mechanics quasifactorization and a more general case which involves covariant derivatives. The admissible classes of electromagnetic fields are described and some illustrative examples are given.

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1. Introduction

In the ordinary one-dimensional supersymmetric quantum mechanics (SUSY QM) [1], a pair of Hamiltonians (superpartners) is factorized as

$$\begin{aligned} H^{(0)} &= -\partial^2 + (\partial\chi)^2 - \partial^2\chi = S^+S^- \qquad \partial \equiv \frac{\mathrm{d}}{\mathrm{d}x} \\ H^{(1)} &= -\partial^2 + (\partial\chi)^2 + \partial^2\chi = S^-S^+ \end{aligned} \tag{1}$$

where

$$S^{\pm} \equiv \mp\partial + \partial\chi \qquad S^+ = (S^-)^{\dagger}. \tag{2}$$

The arbitrary function $\chi(x)$ is called a superpotential. The (not necessarily normalizable) solution of the Schrödinger equation for $H^{(0)}$ with zero energy is $\psi_0^{(0)} = \mathrm{e}^{-\chi}$.

In all cases when one has the factorization relations of the form (1), the supercharge operators $Q^{\pm}$ and the super-Hamiltonian H_S can be introduced [1]:

$$Q^+ \equiv \begin{pmatrix} 0 & 0 \\ S^- & 0 \end{pmatrix} \qquad Q^- \equiv \begin{pmatrix} 0 & S^+ \\ 0 & 0 \end{pmatrix} = (Q^+)^{\dagger} \qquad H_S \equiv \begin{pmatrix} H^{(0)} & 0 \\ 0 & H^{(1)} \end{pmatrix}. \tag{3}$$

The operators $S^{\pm}$ and $H^{(0)}, H^{(1)}$ are called the components of the supercharge and of the super-Hamiltonian, respectively. The relations of the SUSY algebra

$$\{Q^+, Q^-\} = H_S \qquad (Q^+)^2 = (Q^-)^2 = 0 \qquad [Q^{\pm}, H_S] = 0 \tag{4}$$

follow from the definitions (3) and the factorization (1). The last equality in (4) is equivalent to the following intertwining relations:

$$H^{(1)} S^- = S^- H^{(0)} \qquad S^+ H^{(1)} = H^{(0)} S^+. \tag{5}$$

The eigenfunctions of the Hamiltonians are interconnected by the operators $S^{\pm}$:

$$\psi^{(0)}(x; E_n) = E_n^{-1/2} S^+ \psi^{(1)}(x; E_n) \qquad \psi^{(1)}(x; E_n) = E_n^{-1/2} S^- \psi^{(0)}(x; E_n)$$

hence their eigenvalues coincide, except for the zero modes of the supercharges.

Similar to that, in two dimensions the following pair of Hamiltonians can be factorized[1]:

$$H^{(0)} = -\Delta + (\partial_i \chi)^2 - \sigma_3 \Delta \chi = S^+ S^- \tag{6}$$

$$H^{(1)} = -\Delta + (\partial_i \chi)^2 + 2\sigma_1 \partial_1 \partial_3 \chi + \sigma_3 \left(\partial_1^2 - \partial_3^2\right) \chi = S^- S^+$$
$$\partial_i \equiv \frac{\partial}{\partial x_i} \qquad (i = 1, 3) \qquad \Delta \equiv \partial_1^2 + \partial_3^2. \tag{7}$$

This time, the matrix differential operators:

$$S^{\pm} = \mp \partial_1 - \mathrm{i}\sigma_2 \partial_3 + \sigma_1 \partial_3 \chi + \sigma_3 \partial_1 \chi \qquad S^+ = (S^-)^{\dagger} \tag{8}$$

intertwine the Hamiltonians. The superpotential $\chi(\mathbf{x})$ is an arbitrary real scalar function. Note that $H^{(0)}$ is a diagonal Hamiltonian. The solution of its upper component with zero energy is $\psi_0^{(0)} = \mathrm{e}^{-\chi}$.

We remark that it would be natural to generalize the one-dimensional factorization (1) onto the three-dimensional models, similar to (6)–(8). In this case (8) has to contain a term with ∂_2 multiplied by some Pauli matrix (in order to get a factorizable Schrödinger-type Hamiltonian). However, all three Pauli matrices have already been used in (8), so a simple direct three-dimensional generalization of (6)–(8) is impossible. Some generalizations for the three-dimensional case were discussed in [2], where 4×4 Dirac matrices were used, and in [3]. But the construction of the last paper is not adapted to the SUSY diagonalization of the Pauli operator. The three-dimensional factorization of the components of the super-Hamiltonian was considered in [4] leading to the Hamiltonian with terms linear in spatial derivatives (in particular, as a spin–orbital coupling). In this case supercharges with $O(3)$-rotational invariance were chosen. Here we generally do not suppose any spatial symmetry of $S^{\pm}$. Another possible three-dimensional generalization of (6)–(8) will be discussed in section 7.

Returning to the factorization relations (6)–(7) one could again construct the supercharge operators and the super-Hamiltonian[2]:

$$Q^+ \equiv \begin{pmatrix} 0 & 0 \\ S^- & 0 \end{pmatrix} \qquad Q^- \equiv \begin{pmatrix} 0 & S^+ \\ 0 & 0 \end{pmatrix} = (Q^+)^{\dagger}$$
$$H_S = \begin{pmatrix} H^{(0)} & 0 \\ 0 & H^{(1)} \end{pmatrix} \equiv \begin{pmatrix} \underline{\underline{H}}^{(0)} & 0 & 0 \\ 0 & \underline{H}^{(0)} & 0 \\ 0 & 0 & H^{(1)} \end{pmatrix} \tag{9}$$

[1] It will be convenient to denote the two-dimensional vector with components R_1, R_3 as $\mathbf{R}$; $\sigma_i, i = 1, \ldots, 3$, are the Pauli matrices.

[2] In [5] a different definition was used: the positions of the components $\underline{H}^{(0)}$ and $H^{(1)}$ in H_S were interchanged. For readers familiar with the two-dimensional SUSY QM [5, 6] we may add that this corresponds to choosing a different representation of the fermionic creation/annihilation operators.

where

$$\underline{\underline{H}}^{(0)} = -\Delta + (\partial_i \chi)^2 - \Delta\chi \qquad \underline{H}^{(0)} = -\Delta + (\partial_i \chi)^2 + \Delta\chi \qquad i = 1, 3$$

and $H^{(1)}$ is still defined by the first equality of (7). Below we will call $H^{(0)}$ the diagonal component of the super-Hamiltonian and $H^{(1)}$ the matrix one.

From the factorization relations (6)–(7) it follows again that

$$\{Q^+, Q^-\} = H_S \qquad (Q^+)^2 = (Q^-)^2 = 0 \qquad [Q^{\pm}, H_S] = 0. \tag{10}$$

The last equality of (10) will again be equivalent to the intertwining relations:

$$H^{(1)} S^- = S^- H^{(0)} \qquad S^+ H^{(1)} = H^{(0)} S^+. \tag{11}$$

The components of the supercharge operators (8) can be rewritten as

$$S^- = \begin{pmatrix} Q_1^- & P_1^- \\ Q_3^- & P_3^- \end{pmatrix} \qquad S^+ = \begin{pmatrix} Q_1^+ & Q_3^+ \\ P_1^+ & P_3^+ \end{pmatrix} \tag{12}$$

where $(l, m = 1, 3)$

$$Q_l^{\pm} \equiv \mp\partial_l + \partial_l \chi \qquad P_l^{\pm} \equiv \epsilon_{lm} Q_m^{\mp} \qquad \epsilon_{11} = \epsilon_{33} = 0 \qquad \epsilon_{13} = -\epsilon_{31} = 1.$$

If we plug (12) into the intertwining relations (11) we, after some algebra, will recover the usual relations of the two-dimensional SUSY QM [5, 6]:

$$\begin{aligned} H_{lm}^{(1)} Q_m^- &= Q_l^- \underline{\underline{H}}^{(0)} \qquad & Q_l^+ H_{lm}^{(1)} &= \underline{\underline{H}}^{(0)} Q_m^+ \\ H_{lm}^{(1)} P_m^- &= P_l^- \underline{H}^{(0)} \qquad & P_l^+ H_{lm}^{(1)} &= \underline{H}^{(0)} P_m^+ \qquad l, m = 1, 3. \end{aligned} \tag{13}$$

The eigenfunctions of the Hamiltonians $H^{(0)}$, $H^{(1)}$ are interconnected by the operators $S^{\pm}$:

$$\begin{aligned} \psi^{(0)}(\mathbf{x}; E_n) &= E_n^{-1/2} S^+ \psi^{(1)}(\mathbf{x}; E_n) \\ \psi^{(1)}(\mathbf{x}; E_n) &= E_n^{-1/2} S^- \psi^{(0)}(\mathbf{x}; E_n). \end{aligned} \tag{14}$$

It follows that the eigenvalues of the Hamiltonians $H^{(0)}$, $H^{(1)}$ coincide, except for the zero modes of the supercharges. If $H^{(1)}$ is not diagonalizable by constant unitary transformations (see section 3) then the above supersymmetric transformations can simplify the derivation of its spectrum by relating it to the spectrum of a diagonal Hamiltonian $H^{(0)}$ [5, 6]. In the most interesting example [7] one identifies $H^{(1)}$ with the Pauli operator [8] of a non-relativistic spin-1/2 particle in an external electromagnetic field.

The Pauli operator was studied in the framework of SUSY QM in many papers (see [2, 3, 9–13]). In these papers the Pauli operator was usually identified with the *total super-Hamiltonian*, while in the present work the Pauli operator coincides with a *component of the super-Hamiltonian* (in this sense we continue the approach of [7]). An advantage of our approach is the opportunity to consider arbitrary values of the gyromagnetic ratio $g = \frac{2\mu}{e}$ (in contrast to the usual restriction that $g = 2$). Also, we will consider the magnetic fields having *nonzero* components in the (x_1, x_3) plane, for which the Pauli problem is not diagonalizable by rotations. We have to mention that interesting concrete models of the Pauli operator were investigated by means of SUSY QM: a particle with spin in the field of a monopole [2], and a spin-1/2 particle in the magnetic field of a thin straight wire with current [10]. The SUSY diagonalizability of Pauli operators was considered in [12, 13] but only for external fields with definite spatial symmetries.

The paper is organized as follows. In section 2, we consider the simplest case, when the Pauli operator is literally identified with the matrix component of the super-Hamiltonian . The Pauli operator is originally three dimensional but admits the separation of the x_2 variable. We derive the expressions for the magnetic field and the scalar potential in the Pauli operator and for the superpotential that follows from such an identification.

In section 3 we enlarge the class of the Pauli operators, which can be identified with the matrix component of the super-Hamiltonian, by allowing constant unitary rotations, and describe the class of superpotentials for which it is possible. The content of sections 2 and 3 should be considered as a generalization of [7] for the case of nonzero electric current.

In section 4, we investigate the case of a charged particle in detail. It turns out that the separation of the x_2 variable is not trivial. The class of magnetic fields for which the Pauli operator of a charged particle can be identified (up to a unitary rotation) with $H^{(1)}$ is described. It is smaller than that for a neutral particle.

In section 5, we describe a new generalization of the SUSY QM transformations. It involves the transition from the usual derivatives to the covariant ones both in the components of the supercharge operators and those of the super-Hamiltonian. We also prove the general statement that the quasifactorization relations [5, 6] always describe a pair of factorized Hamiltonians, one of which is diagonal.

In section 6, we use these generalized transformations to diagonalize the Pauli operators for more general configurations of magnetic field in the case when the gyromagnetic ratio is equal to 2. In particular, the component B_2 of the magnetic field along the x_2 direction can be arbitrary. Examples of the Pauli operators allowing the above treatment are given at the end of sections 2, 4 and 6.

In section 7, we construct a three-dimensional generalization of the two-dimensional factorization relations described in the two previous sections.

2. Pauli Hamiltonian as a matrix component of the super-Hamiltonian

A non-relativistic particle with spin 1/2 in an external electromagnetic field is described [8] by the Pauli Hamiltonian[3]:

$$H_P = -\vec{D}^2 - \mu\vec{\sigma}\cdot\vec{B}(\vec{x}) + U(\vec{x}) \qquad (\hbar = c = 2m = 1) \qquad \vec{D} \equiv \vec{\partial} - \mathrm{i}e\vec{A}(\vec{x}) \tag{15}$$

where $\vec{A}(\vec{x})$ is the electromagnetic vector potential; $\vec{B}(\vec{x}) = \mathrm{rot}\,\vec{A}(\vec{x})$ is the external magnetic field; $U(\vec{x})$ is the external scalar potential (not necessarily an electrostatic one); e, μ are the charge and the magnetic moment[4] of the particle. The operator (15) acts on the two-component wavefunctions.

Let us assume that all external fields in (15) depend on two coordinates (for example $(x_1, x_3) \equiv \mathbf{x}$) only. Then the wavefunction of a particle that moves freely along the x_2 direction can be written as

$$\psi(\vec{x}) = \mathrm{e}^{-\mathrm{i}kx_2}\psi(\mathbf{x}) \tag{16}$$

and the Pauli Hamiltonian, acting on $\psi(\mathbf{x})$, assumes the form:

$$H_P = -\mathbf{D}^2 + (k + eA_2(\mathbf{x}))^2 - \mu\vec{\sigma}\cdot\vec{B}(\mathbf{x}) + U(\mathbf{x}). \tag{17}$$

Insofar as the operator (17) is a 2×2 matrix differential operator in the two-dimensional space (x_1, x_3), one can try to identify it with the matrix component $H^{(1)}$ of the super-Hamiltonian:

$$H_P = H^{(1)} + E_0.$$

This idea was set forth in [7], but in that paper it was additionally assumed that the magnetic field has no sources: $\vec{j} = \vec{0}$. In this paper, in contrast, we consider the most general case of

[3] From this moment on, $\vec{R}$ denotes a three-dimensional vector with the components R_1, R_2, R_3.

[4] We stress that the value of the gyromagnetic ratio $g = 2\mu/e$ can be *arbitrary*.

the equivalence of H_P and $H^{(1)}$, when the current density can be nonzero. Thus one can try to consider a wider class of Pauli operators.

If such an identification is possible, the corresponding Pauli operator can be diagonalized by the supersymmetric transformations from section 1, and its spectrum is the sum of the spectra of $\underline{H}^{(0)}$ and $\underline{\underline{H}}^{(0)}$ (see (13) and (14)). It is possible only if the following relations between the external fields in the Pauli operator and the superpotential $\chi(\mathbf{x})$ are satisfied:

$$-\mu\vec{B} = \left(2\partial_1\partial_3\chi, 0, \left(\partial_1^2 - \partial_3^2\right)\chi\right) \tag{18}$$

$$U(\mathbf{x}) = (\partial_i\chi)^2 - (k + eA_2)^2 + E_0 \qquad i = 1, 3. \tag{19}$$

From the absence of first derivatives in $H^{(1)}$, and, consequently, in H_P, it follows that either $e = 0$ (a neutral particle) or $A_1 = A_3 = 0$.

In equation (19) all terms[5], except $(k + eA_2)^2$ and possibly E_0, have no dependence on k. The k-dependent parts of these two terms cancel only if $e = 0$ and $E_0 = k^2$, therefore we assume this choice of e and E_0 in the remaining part of this section. From the Maxwell equation $\partial_i B_i = 0$ and (18) one can infer the following constraint onto the superpotential $\chi(\mathbf{x})$:

$$\partial_3\left(3\partial_1^2 - \partial_3^2\right)\chi = 0. \tag{20}$$

The general solution of this equation has the form (see the appendix):

$$\chi(\mathbf{x}) = F(x_1) + G(-x_1/2 + (\sqrt{3}/2)x_3) + H(-x_1/2 - (\sqrt{3}/2)x_3) \tag{21}$$

where F, G, H are arbitrary thrice-differentiable functions of their arguments.

Equalities (18) and (19) allow us to express all the physical quantities in terms of these functions. In particular,

$$B_1(\mathbf{x}) = \frac{\sqrt{3}}{2\mu}(G'' - H'') \qquad B_3(\mathbf{x}) = \frac{1}{\mu}(-F'' + 1/2(G'' + H'')) \tag{22}$$

and the electric current $\vec{j}(\mathbf{x}) = \mathrm{rot}\,\vec{B}(\mathbf{x})$ has, obviously, only one nonzero component:

$$j_2(\mathbf{x}) = \frac{1}{4\pi\mu}[F''' + G''' + H'''].$$

The scalar potential (19) then can also be expressed in terms of F, G, H:

$$U(\mathbf{x}) = F'^2 + G'^2 + H'^2 - F'G' - F'H' - G'H'. \tag{23}$$

Thus the Pauli operator with the external fields as in (22) and (23) coincides with the Hamiltonian $H^{(1)}$ and can be diagonalized by the supersymmetric transformations. One can check that in the case $j_2 = 0$ our solution for $\chi(\mathbf{x})$ coincides with that offered in [7] for the case of a neutral particle.

Example. Let us choose specific functions F, G, H in (21):

$$F(x) = G(x) = H(x) = ax^3$$

where a is an arbitrary constant. Then, from (22),

$$B_1 = -\frac{9a}{\mu}x_3 \qquad B_3 = \frac{9a}{\mu}x_1.$$

[5] In [7], A_2 was chosen such that $k + eA_2$ is k independent. We will not use this idea because A_2 is a quantity that enters in the original three-dimensional Pauli operator (15) and therefore cannot depend on k. In the following sections we will discuss some cases when χ depends on k, but for now we assume that it is k independent.

The electric current that generates this system of external fields is homogeneous and is directed along the axis x_2:

$$j_2 = -\frac{9a}{2\pi\mu}.$$

The scalar potential (23) in H_P depends on $\rho^2 \equiv x_1^2 + x_3^2$ only:

$$U(\mathbf{x}) = (\partial_i\chi)^2 = \gamma\rho^4 \qquad \gamma = \frac{81a^2}{16}$$

and the diagonal component of the super-Hamiltonian is proportional to the unit matrix:

$$\underline{\underline{H}}^{(0)} = \underline{H}^{(0)} = -\Delta + \gamma\rho^4$$

since $\Delta\chi = 0$ in this case. Thus the spectrum of the Pauli operator in question consists of twofold degenerate levels of the anharmonic oscillator $\gamma\rho^4$.

3. Unitary rotation

In order to enlarge the class of external fields for which the Pauli Hamiltonian can be diagonalized by the supersymmetric factorization, one can identify H_P with the operator, which is unitarily equivalent to the matrix component of the super-Hamiltonian [7]:

$$H_P = \tilde{H}^{(1)} + E_0 \qquad \tilde{H}^{(1)} = \mathcal{U}H^{(1)}\mathcal{U}^+$$

where $\mathcal{U}$ is a constant 2×2 matrix $\mathcal{U} = \alpha_0 + \mathrm{i}\vec{\alpha}\vec{\sigma}$; $\alpha_0^2 + \vec{\alpha}^2 = 1$; α_0, α_j are real constants; $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

Such rotation obviously does not change the spectrum of the matrix operator $H^{(1)}$, but leads to additional freedom because of the presence of the new parameters α_0, α_j. Now the relation between the components of the magnetic field and the superpotential χ is more complicated compared to (18):

$$\mu B_i = \left[\left(\alpha_0^2 - \vec{\alpha}^2\right)\delta_{ik} + 2(\alpha_j\alpha_0\epsilon_{ijk} + \alpha_k\alpha_i)\right]X_k \qquad (i, k = 1, 2, 3) \tag{24}$$

where $\vec{X} \equiv \left(-2\partial_1\partial_3\chi, 0, \left(\partial_3^2 - \partial_1^2\right)\chi\right)$. The scalar potential is still defined by (19). Taking the divergence of equation (24), we obtain

$$\begin{aligned}0 = \mu\partial_i B_i &= \partial_i\left[\left(\alpha_0^2 - \vec{\alpha}^2\right)\delta_{ik} + 2(\alpha_j\alpha_0\epsilon_{ijk} + \alpha_k\alpha_i)\right]X_k \\ &= -\left(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2\right)2\partial_1^2\partial_3\chi + \left(\alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2\right)\left(\partial_3^3 - \partial_1^2\partial_3\right)\chi \\ &\quad + (2\alpha_1\alpha_3 - 2\alpha_2\alpha_0)\left(\partial_1\partial_3^2 - \partial_1^3\right)\chi - (2\alpha_2\alpha_0 + 2\alpha_1\alpha_3)2\partial_1\partial_3^2\chi.\end{aligned}$$

Hence, the superpotential χ satisfies the following partial differential equation with constant coefficients:

$$\begin{aligned}&\left[2(\alpha_2\alpha_0 - \alpha_1\alpha_3)\partial_1^3 - \left(3\alpha_0^2 + \alpha_1^2 - 3\alpha_2^2 - \alpha_3^2\right)\partial_1^2\partial_3 - 2(\alpha_1\alpha_3 + 3\alpha_2\alpha_0)\partial_1\partial_3^2 \right.\\ &\qquad \left. + \left(\alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2\right)\partial_3^3\right]\chi = 0.\end{aligned} \tag{25}$$

Equation (20) is a partial case of (25) with $\alpha_0 = 1$; $\alpha_i = 0$.

It is shown in the appendix that the superpotentials χ, satisfying this equation and therefore allowing the identification $H_P = \tilde{H}^{(1)} + E_0$, must have the form[6]:

$$\chi = F(t_1x_3 + x_1) + G(t_2x_3 + x_1) + H(t_3x_3 + x_1) \tag{26}$$

[6] If some of t_i are complex one can take only such functions F, G, H that the superpotential is a real function.

where t_1, t_2, t_3 are the roots of the characteristic polynomial, which is determined by the operator in (25):

$$\begin{aligned}\big(2(\alpha_2\alpha_0-\alpha_1\alpha_3)\partial_1^3-\big(3\alpha_0^2+\alpha_1^2-3\alpha_2^2-\alpha_3^2\big)\partial_1^2\partial_3-2(\alpha_1\alpha_3+3\alpha_2\alpha_0)\partial_1\partial_3^2\\ +2\big(\alpha_0^2-\alpha_1^2-\alpha_2^2+\alpha_3^2\big)\partial_3^3\big)=c_0(\partial_3-t_1\partial_1)(\partial_3-t_2\partial_1)(\partial_3-t_3\partial_1)\end{aligned} \quad (27)$$

$c_0 = 2\big(\alpha_0^2-\alpha_1^2-\alpha_2^2+\alpha_3^2\big)$.

In the case of a neutral particle ($e = 0$ but $\mu \neq 0$) any choice of α_i and the corresponding superpotential (26) leads to a Pauli operator determined by (17), (19) and (24), which is diagonalizable by supersymmetric transformations.

For a charged particle ($e \neq 0$) there are stronger restrictions: as we mentioned in section 2, this approach is valid only for $A_1 = A_3 = 0$, and, consequently, $B_2 = 0$ and $j_1 = j_3 = 0$. Then, from (24),

$$(2\alpha_3\alpha_0+2\alpha_1\alpha_2)B_1(\mathbf{x})+(2\alpha_3\alpha_2-2\alpha_1\alpha_0)B_3(\mathbf{x})=0. \quad (28)$$

If either of the constants in (28) is nonzero, the Pauli operator can be trivially diagonalized by a constant unitary rotation, after which $\vec{B}$ is directed along x_3. Hence, the interesting situation is when $2\alpha_3\alpha_0 + 2\alpha_1\alpha_2 = 2\alpha_3\alpha_2 - 2\alpha_1\alpha_0 = 0$. One can check that it is possible only in the following two cases:

(1) $\alpha_3 = \alpha_1 = 0$
(2) $\alpha_2 = \alpha_0 = 0$.

Unfortunately, in the variant (1) we could not find solutions that satisfy (19) with the scalar and vector potentials U and $\vec{A}$ independent of k.

In the variant (2) the roots t_1, t_2, t_3 (27) have the form

$$t_1=\frac{2\alpha_1\alpha_3}{\alpha_3^2-\alpha_1^2}\qquad t_2=\mathrm{i}\qquad t_3=-\mathrm{i}.$$

Hence, taking the reality of the superpotential into account, (26) leads to

$$\chi=F\left(x_1+\frac{2\alpha_1\alpha_3}{\alpha_3^2-\alpha_1^2}x_3\right)+\operatorname{Re}g(z)\qquad z=x_1+\mathrm{i}x_3 \quad (29)$$

where $g(z) = 2G(z)$ is an arbitrary holomorphic function of z.

4. The dependence of the spectrum on the momentum along the axis x_2

It can be shown that in the case of a charged particle ($e \neq 0$) the superpotential (29) can satisfy (19) with the scalar and vector potentials U and $\vec{A} = (0, A_2, 0)$ *independent of* k only if $F = 0$:

$$\chi=\operatorname{Re}g(z). \quad (30)$$

Then (24) has the following solution[7]:

$$B_1=-\frac{2}{\mu}\operatorname{Im}g''\qquad B_3=-\frac{2}{\mu}\operatorname{Re}g''\qquad A_2=-\frac{2}{\mu}\operatorname{Re}g'+C. \quad (31)$$

Let us choose

$$g(z)=g_0(z)-\frac{2e}{\mu}kz\qquad C=C_0-\frac{4e}{\mu^2}k \quad (32)$$

[7] A similar expression for the magnetic field was found in [7]. However in this paper we allow $j_2 = \partial_1 B_3 - \partial_3 B_1$ to be nonzero (the other two components of the current density are zero because of our assumptions). This is possible if $g_0(z)$ has singularities, i.e., it is not an entire function (see the example below). Another difference of the approach of [7] is that it did not allow the dependence of energy levels on k to be obtained in contrast to ours.

where neither g_0 nor C_0 depend on k. Then from (31)

$$B_1 = -\frac{2}{\mu}\,\mathrm{Im}\, g_0'' \qquad B_3 = -\frac{2}{\mu}\,\mathrm{Re}\, g_0'' \qquad A_2 = -\frac{2}{\mu}\,\mathrm{Re}\, g_0' + C_0. \tag{33}$$

Note that A_2 does not depend on k. Now let us prove that U can also be made k independent. Plugging (30), (32) and (33) into (19), we can check that

$$U = |g_0'|^2 - \left(\frac{2e}{\mu}\,\mathrm{Re}\, g_0' - C_0 e\right)^2 + E_0 - \left(1 - \frac{4e^2}{\mu^2}\right)k^2 - 2C_0 ek. \tag{34}$$

By choosing

$$E_0 = \left(1 - \frac{4e^2}{\mu^2}\right)k^2 + 2C_0 ek \tag{35}$$

we can make U k-independent. Indeed, from (34)

$$U = |g_0'|^2 - \left(\frac{2e}{\mu}\,\mathrm{Re}\, g_0' - C_0 e\right)^2. \tag{36}$$

Plugging (30) and (32) into (6), we see that the Pauli operator with external fields (33) and (36) is equivalent to the diagonal Hamiltonian

$$H^{(0)} = -\Delta + \left|g_0' - \frac{2e}{\mu}k\right|^2 - \sigma_3 \Delta\,\mathrm{Re}\, g_0. \tag{37}$$

Note that k enters into $H^{(0)}$ nontrivially. The last term in $H^{(0)}$ is zero everywhere except the points where g has singularities.

Example. Let us choose a specific function $g_0(z)$ in (32):

$$g_0(z) = a \ln z$$

with a being a real constant. Then we readily obtain from the corresponding general expressions that

$$A_2 = -\frac{2ax_1}{\mu\rho^2} + C_0 \qquad B_1 = -\frac{4a}{\mu}\frac{x_1x_3}{\rho^4} \qquad B_3 = \frac{2a}{\mu}\frac{x_1^2 - x_3^2}{\rho^4} \qquad j_2 = -\frac{a}{\mu}\partial_1\delta(x_1)\delta(x_3).$$

This electric current density can be interpreted as a 'dipole current'—two thin linear currents in opposite directions, the distance between them being small.

In this case, (36) turns into

$$U(\mathbf{x}) = \frac{a^2}{\rho^2} - \left(\frac{aex_1}{\mu\rho^2} - C_0 e\right)^2.$$

From (37) it follows that the spectrum of our Pauli operator coincides with the spectrum of the diagonal Hamiltonian $H^{(0)}$, which has the form:

$$H^{(0)} = -\partial^2 + \left|\frac{a}{z} - \frac{2ek}{\mu}\right|^2 - 2\pi a\delta(x_1)\delta(x_3).$$

5. More general factorization method

In the Pauli operator (17) each derivative enters the covariant derivative combination $D_j \equiv \partial_j - \mathrm{i}eA_j$. Therefore, it seems natural to use the covariant derivatives in the operators (6)–(8) too[8]:

$$S^{\pm} = \mp D_1 - \mathrm{i}\sigma_2 D_3 + \sigma_1 \partial_3 \chi + \sigma_3 \partial_1 \chi \qquad S^+ = (S^-)^{\dagger}. \tag{38}$$

The so-defined intertwining operators generate the following pair of factorized Hamiltonians:

$$H^{(0)} = S^+ S^- = -\mathbf{D}^2 + (\partial_i \chi)^2 - e\sigma_2 B_2 - \sigma_3 \Delta \chi \tag{39}$$

$$\begin{aligned} H^{(1)} &= S^- S^+ = -\mathbf{D}^2 + (\partial_i \chi)^2 + e\sigma_2 B_2 + 2\sigma_1 \partial_1 \partial_3 \chi + \sigma_3 \big(\partial_1^2 - \partial_3^2\big)\chi \\ B_2 &\equiv \partial_3 A_1 - \partial_1 A_3 \qquad i = 1, 3. \end{aligned} \tag{40}$$

The Hamiltonians (7) obey the intertwining equations similar to (11):

$$H^{(1)} S^- = S^- H^{(0)} \qquad S^+ H^{(1)} = H^{(0)} S^+$$

and their eigenfunctions are interconnected.

Similar to sections 2 and 3 one can try to identify $H^{(1)}$ with a Pauli operator up to constant unitary rotations:

$$H_P = \tilde{H}^{(1)} + E_0 \qquad \tilde{H}^{(1)} = \mathcal{U} H^{(1)} \mathcal{U}^+ \tag{41}$$

and $H^{(0)}$ is unitary equivalent to a diagonal Hamiltonian:

$$H^{(0)} = \mathcal{V}^+ \tilde{H}^{(0)} \mathcal{V} \tag{42}$$

$$\tilde{H}^{(0)} = \begin{pmatrix} \underline{\underline{\tilde{H}}}^{(0)} & 0 \\ 0 & \underline{\tilde{H}}^{(0)} \end{pmatrix}. \tag{43}$$

If such an identification is possible, the corresponding Pauli operator can be diagonalized by the above supersymmetric-like transformations and its spectrum is the sum of the spectra of $\underline{\tilde{H}}^{(0)}$ and $\underline{\underline{\tilde{H}}}^{(0)}$. Below we shall show that the more general factorization method presented in this section allows us to diagonalize a broader class of Pauli operators than the usual supersymmetric methods described in the previous sections.

From (39)–(42) one may infer the new factorization relations:

$$\tilde{H}^{(0)} = \tilde{S}^+ \tilde{S}^- \qquad \tilde{H}^{(1)} = \tilde{S}^- \tilde{S}^+ \qquad \tilde{S}^- \equiv \mathcal{U} S^- \mathcal{V}^+ \qquad \tilde{S}^+ \equiv \mathcal{V} S^+ \mathcal{U}^+ = (\tilde{S}^-)^{\dagger}. \tag{44}$$

Note that from (44) we can construct the SUSY algebra relations, similar to (3)–(4), or (9)–(10), and thus obtain a new realization of SUSY QM in two dimensions, which differs from that given in [5, 6].

From (44) the following intertwining relations can be derived:

$$\tilde{H}^{(1)} \tilde{S}^- = \tilde{S}^- \tilde{H}^{(0)} \qquad \tilde{S}^+ \tilde{H}^{(1)} = \tilde{H}^{(0)} \tilde{S}^+$$

and the eigenfunctions of $\tilde{H}^{(0)}$, $\tilde{H}^{(1)}$ are interconnected by the operators $\tilde{S}^{\pm}$:

$$\tilde{\psi}^{(0)}(\mathbf{x}; E_n) = E_n^{-1/2} \tilde{S}^+ \tilde{\psi}^{(1)}(\mathbf{x}; E_n)$$
$$\tilde{\psi}^{(1)}(\mathbf{x}; E_n) = E_n^{-1/2} \tilde{S}^- \tilde{\psi}^{(0)}(\mathbf{x}; E_n).$$

Thus the eigenvalues of the Hamiltonians $\tilde{H}^{(0)}$, $\tilde{H}^{(1)}$ again coincide up to the zero modes of the supercharge components $\tilde{S}^{\pm}$.

[8] Note that in the simplest particular case $\chi = 0$ the operators (38) can be reduced to the SUSY generators given in [3, 9] by the multiplication onto σ_1 or σ_3 and a redefinition of axes.

Now let us parametrize the matrices $\tilde{S}^{\pm}$ in the following way, which is similar to (12):

$$\tilde{S}^{-} = \begin{pmatrix} \tilde{Q}_1^- & \tilde{P}_1^- \\ \tilde{Q}_3^- & \tilde{P}_3^- \end{pmatrix} \qquad \tilde{S}^{+} = \begin{pmatrix} \tilde{Q}_1^+ & \tilde{Q}_3^+ \\ \tilde{P}_1^+ & \tilde{P}_3^+ \end{pmatrix}. \tag{45}$$

Plugging these operators into (44) and (43), one can infer the following quasifactorization relations:

$$\begin{gathered} \underline{\underline{\tilde{H}}}^{(0)} = \tilde{Q}_l^+ \tilde{Q}_l^- \qquad \underline{\underline{\tilde{H}}}^{(0)} = \tilde{P}_l^+ \tilde{P}_l^- \qquad \tilde{H}_{lm}^{(1)} = \tilde{Q}_l^- \tilde{Q}_m^+ + \tilde{P}_l^- \tilde{P}_m^+ \\ \tilde{P}_l^+ \tilde{Q}_l^- = \tilde{Q}_l^+ \tilde{P}_l^- = 0 \qquad l, m = 1, 3. \end{gathered} \tag{46}$$

From them we can deduce an analogue of the intertwining relations (13):

$$\begin{gathered} \tilde{H}_{lm}^{(1)} \tilde{Q}_m^- = \tilde{Q}_l^- \underline{\underline{\tilde{H}}}^{(0)} \qquad \tilde{Q}_l^+ \tilde{H}_{lm}^{(1)} = \underline{\underline{\tilde{H}}}^{(0)} \tilde{Q}_m^+ \\ \tilde{H}_{lm}^{(1)} \tilde{P}_m^- = \tilde{P}_l^- \underline{\tilde{H}}^{(0)} \qquad \tilde{P}_l^+ \tilde{H}_{lm}^{(1)} = \underline{\tilde{H}}^{(0)} \tilde{P}_m^+ \qquad l, m = 1, 3. \end{gathered} \tag{47}$$

One may also note that $\tilde{H}^{(1)}$ can be decomposed as

$$\tilde{H}_{lm}^{(1)} = \underline{\tilde{H}}_{lm}^{(1)} + \underline{\underline{\tilde{H}}}_{lm}^{(1)} \qquad \underline{\tilde{H}}_{lm}^{(1)} \equiv \tilde{Q}_l^- \tilde{Q}_m^+ \qquad \underline{\underline{\tilde{H}}}_{lm}^{(1)} \equiv \tilde{P}_l^- \tilde{P}_m^+ \tag{48}$$

such that $\underline{\tilde{H}}_{lm}^{(1)} \underline{\underline{\tilde{H}}}_{mn}^{(1)} = \underline{\underline{\tilde{H}}}_{lm}^{(1)} \underline{\tilde{H}}_{mn}^{(1)} = 0$, and relations (47) can be rewritten in the form:

$$\begin{gathered} \underline{\tilde{H}}_{lm}^{(1)} \tilde{Q}_m^- = \tilde{Q}_l^- \underline{\underline{\tilde{H}}}^{(0)} \qquad \tilde{Q}_l^+ \underline{\tilde{H}}_{lm}^{(1)} = \underline{\underline{\tilde{H}}}^{(0)} \tilde{Q}_m^+ \\ \underline{\underline{\tilde{H}}}_{lm}^{(1)} \tilde{P}_m^- = \tilde{P}_l^- \underline{\tilde{H}}^{(0)} \qquad \tilde{P}_l^+ \underline{\underline{\tilde{H}}}_{lm}^{(1)} = \underline{\tilde{H}}^{(0)} \tilde{P}_m^+ \qquad l, m = 1, 3. \end{gathered} \tag{49}$$

Relations (46)–(49) have exactly the same form as the well-known SUSY QM quasifactorization relations and the SUSY QM intertwining relations [5, 6]. The former are reduced to the latter in the case $\mathcal{U} = \mathcal{V} = I$ and $A_1 = A_3 = 0$.

From the above algebra we see that *any* quasifactorization relations in fact describe a pair of *factorized* (in the sense of (44)) Hamiltonians, one of which is diagonal. This is a rather general statement, because relations (46)–(49) follow from the definitions (43)–(45), no matter what form the components of the supercharge $\tilde{Q}_l^{\pm}$, $\tilde{P}_l^{\pm}$ have.

6. The cases when $H^{(0)}$ is diagonalizable

Equation (41) is equivalent to the following pair of equations:

$$\mu B_i = \left[\left(\alpha_0^2 - \vec{\alpha}^2\right) \delta_{ik} + 2(\alpha_j \alpha_0 \epsilon_{ijk} + \alpha_k \alpha_i)\right] Y_k \qquad (i, k = 1, 2, 3) \tag{50}$$

$$U(\mathbf{x}) = (\partial \chi)^2 - (k + eA_2)^2 + E_0 \tag{51}$$

where $\vec{Y} \equiv \left(-2\partial_1 \partial_3 \chi, -eB_2, -\left(\partial_1^2 - \partial_3^2\right)\chi\right)$.

Let us consider for simplicity the case when the gyromagnetic ratio $2\mu/e$ is equal to[9] $g = 2$ and $\alpha_0 = \alpha_2 = 0$. Then (50) is reduced to (24), which is solved in section 3, the solution being given by (26). Assume further that $\alpha_3 = 1$; $\alpha_1 = 0$, i.e., $\mathcal{U} = \sigma_3$. Then, equation (26) is reduced to a partial case of (29):

$$\chi = F(x_1) + \operatorname{Re} g(z) \qquad z = x_1 + \mathrm{i} x_3. \tag{52}$$

[9] This is the first instance when we fix the gyromagnetic ratio to the same value $g = 2$ that appeared in [3, 9]. However, in this paper we consider a much larger class of the configurations of external fields compared to [3, 9], because, in contrast to them, we allow the magnetic field component in the (x_1, x_3) plane to be nonzero.

Plugging the superpotential (52) into (50) we obtain

$$B_1 = -\frac{2}{e}\,\mathrm{Im}\,g'' \qquad B_3 = -\frac{2}{e}\,\mathrm{Re}\,g'' - \frac{1}{e}F'' \qquad A_2 = -\frac{2}{e}\,\mathrm{Re}\,g' - \frac{1}{e}F' + \text{const}.$$

The first component $H^{(0)}$ of the super-Hamiltonian is unitarily equivalent to a diagonal Hamiltonian in the following two[10] cases:

(a) there is some constant C_B such that $B_2 = C_B \Delta\chi$;
(b) $\Delta\chi = 0$.

Unfortunately, in case (a) it is impossible to satisfy (51) when both U and A_2 are k independent. In the rest of this section we will consider case (b). Then the Hamiltonian $H^{(0)}$ is already diagonal; therefore, in the rest of the paper we will assume that $\mathcal{V} = I$.

If the superpotential χ (52) satisfies the condition $\Delta\chi = 0$, it follows that $F(x_1)$ is a linear function. One can check that we can set $F(x_1) = 0$ without narrowing the class of Pauli operators that can be diagonalized by our factorization method. So,

$$\chi = \mathrm{Re}\,g(z). \tag{53}$$

This is the same superpotential as in (30). However, this time $g(z)$ cannot contain singularities anywhere except at infinity, because otherwise $\Delta\chi$ would be nonzero.

The magnetic field components B_1 and B_3 are obtained from the same equation as (24), therefore they will have the form[11] (31):

$$B_1 = -\frac{2}{e}\,\mathrm{Im}\,g'' \qquad B_3 = -\frac{2}{e}\,\mathrm{Re}\,g'' \qquad A_2 = -\frac{2}{e}\,\mathrm{Re}\,g' + C.$$

The component B_2 can be arbitrary. Similar to section 4, we choose the superpotential and the constant in the magnetic potential in the form (32):

$$g(z) = g_0(z) - 2kz \qquad C = C_0 - \frac{4}{e^2}k \tag{54}$$

Then the magnetic field components B_1 and B_3 take the form (33):

$$B_1 = -\frac{2}{e}\,\mathrm{Im}\,g_0'' \qquad B_3 = -\frac{2}{e}\,\mathrm{Re}\,g_0'' \tag{55}$$

while B_2 remains arbitrary. We get the scalar field $U(\mathbf{x})$ from (51), which coincides with (19). If we assume that E_0 has the form (35):

$$E_0 = -3k^2 + 2C_0ek$$

then the scalar field will be defined by (36)

$$U = |g_0'|^2 - (2\,\mathrm{Re}\,g_0' - C_0e)^2$$

Thus, as in section 4, all physical fields in the Pauli operator are k independent.

Plugging (53) and (54) into (39), we see that the spectrum of the Pauli operator coincides with the spectrum of the Hamiltonian

$$H^{(0)} = -\mathbf{D}^2 - \sigma_2 B_2 + |g_0' - 2k|^2.$$

After a unitary rotation by $\mathcal{V} = (\sigma_2 + \sigma_3)/\sqrt{2}$, $H^{(0)}$ will turn into a diagonal Hamiltonian analogous to (37):

$$\tilde{H}^{(0)} = -\mathbf{D}^2 - \sigma_3 B_2 + |g_0' - 2k|^2. \tag{56}$$

[10] We do not consider the case when $A_1 = A_3 = 0$ that has already been investigated above.
[11] We keep taking into account that the gyromagnetic ratio $2\mu/e$ is fixed at 2.

Example. Let us choose a specific function $g(z)$ in (53):

$$g(z)=\frac{\omega}{6}z^3.$$

Then we readily obtain from the corresponding general expressions (55)–(56) that

$$\vec{B}=-\frac{2\omega}{e}(x_3,0,x_1)+(0,B_2,0) \qquad \vec{j}=\vec{0}$$

$$U=\frac{\omega^2}{4}\rho^4-\left(w\left(x_1^2-x_3^2\right)+\text{const}\right)^2 \qquad \tilde{H}^{(0)}=-\mathbf{D}^2-\sigma_3B_2+\left|\frac{\omega}{2}z^2-2k\right|^2.$$

7. A three-dimensional factorization

In this section, we will show that the above factorization method allows a three-dimensional generalization. It is realized by the operators

$$R^{\pm}\equiv\mp\vec{\sigma}\cdot\vec{D}+V_0(\vec{x}) \qquad R^+=(R^-)^\dagger \tag{57}$$

that can be used for the factorization[12] of the following pair of Hamiltonians:

$$\check{H}^{(0)}\equiv R^+R^-=-\vec{D}^2-\vec{\sigma}\cdot(\vec{\partial}V_0+e\vec{B})+V_0^2 \tag{58}$$

$$\check{H}^{(1)}\equiv R^-R^+=-\vec{D}^2+\vec{\sigma}\cdot(\vec{\partial}V_0-e\vec{B})+V_0^2 \tag{59}$$

where again $\vec{D}=\vec{\partial}-\mathrm{i}e\vec{A}$; $\vec{B}=\mathrm{rot}\,\vec{A}$; $V_0(\vec{x})$ is an arbitrary real scalar function.

The operators (58)–(59) can then be treated in the same way as in section 5: $\check{H}^{(0)}$ can be made unitarily equivalent to a Pauli operator in three dimensions (15):

$$H_P=\tilde{\check{H}}^{(1)}+E_0 \qquad \tilde{\check{H}}^{(1)}=\check{\mathcal{U}}\check{H}^{(1)}\check{\mathcal{U}}^+ \tag{60}$$

and $\check{H}^{(0)}$ to a diagonal Hamiltonian $\tilde{\check{H}}^{(0)}$:

$$\check{H}^{(0)}=\check{\mathcal{V}}^+\tilde{\check{H}}^{(0)}\check{\mathcal{V}}. \tag{61}$$

Now let us prove that the operators (57)–(59) are indeed a generalization of (38)–(40). Namely, assume that V_0 and $\vec{A}$ depend on two coordinates (x_1,x_3) only. Then, similar to section 2, the wavefunction of a particle that moves freely along the x_2 direction has the form (16), and the operators (57)–(59), acting on $\psi(\mathbf{x})$, assume the form:

$$R^{\pm}\to\hat{R}^{\pm}=\mp[\sigma_1D_1+\sigma_3D_3-\mathrm{i}(k+eA_2)\sigma_2]+V_0 \tag{62}$$

$$\check{H}^{(0)}\to\hat{H}^{(0)}=-\mathbf{D}^2+(k+eA_2)^2-\vec{\sigma}\cdot(\vec{\partial}V_0+e\vec{B})+V_0^2=\hat{R}^+\hat{R}^- \tag{63}$$

$$\check{H}^{(1)}\to\hat{H}^{(1)}=-\mathbf{D}^2+(k+eA_2)^2+\vec{\sigma}\cdot(\vec{\partial}V_0-e\vec{B})+V_0^2=\hat{R}^-\hat{R}^+. \tag{64}$$

The Pauli operator and the diagonal Hamiltonian in (60) and (61) will also transform to two-dimensional operators.

Now, if we assume that $V_0=\partial_3\chi$ and $A_2=\frac{1}{e}(\partial_1\chi-k)$ then one can check that (62)–(64) can be rewritten in terms of the operators from section 5:

$$\hat{R}^+=\sigma_1S^+ \qquad \hat{R}^-=S^-\sigma_1 \qquad \hat{H}^{(0)}=H^{(0)} \qquad \hat{H}^{(1)}=\sigma_1H^{(1)}\sigma_1$$

where $S^{\pm}$ are defined in (38), and $\tilde{H}^{(0)}$, $\tilde{H}^{(1)}$ in (39), (40). Then one can check that the two-dimensional versions of (60) and (61) are equivalent to (41) and (42), if we set[13]:

$$\check{\mathcal{U}}=\mathcal{U}\sigma_1 \qquad \check{\mathcal{V}}=\mathcal{V}.$$

Thus, we have proved that the operators (57)–(59) are indeed a generalization of (38)–(40). Such a generalization may allow one to diagonalize a more general class of three-dimensional Pauli operators.

[12] The factorization (57)–(59) was proposed in [14] for the study of massless Dirac operators in Euclidean spacetime.
[13] Note that the choice of the unitary matrices $\mathcal{V}=I;\mathcal{U}=\sigma_2$, which we have widely used in sections 3–6, is thus equivalent to the simplest choice in the new formalism of this section: $\check{\mathcal{V}}=\check{\mathcal{U}}=I$.

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Appendix

It is straightforward that any differential operator of the form

$$c_0\partial_3^3 + c_1\partial_3^2\partial_1 + c_2\partial_3\partial_1^2 + c_3\partial_1^3$$

with $c_0 \neq 0$ can be factorized into a product of differential operators of first order:

$$c_0(\partial_3 - t_1\partial_1)(\partial_3 - t_2\partial_1)(\partial_3 - t_3\partial_1)$$

where t_1, t_2, t_3 are the roots of the polynomial $c_0x^3 + c_1x^2 + c_2x + c_3$. Assume for simplicity that these roots are different. Then the following statement is true: if

$$(\partial_3 - t_1\partial_1)(\partial_3 - t_2\partial_1)(\partial_3 - t_3\partial_1)f = 0 \tag{A1}$$

then $f = F(t_1x_3 + x_1) + G(t_2x_3 + x_1) + H(t_3x_3 + x_1)$.

Proof. If (A1) is satisfied, then there exists a function $f_0(t_1x_3 + x_1)$ such that

$$(\partial_3 - t_2\partial_1)(\partial_3 - t_3\partial_1)f = f_0(t_1x_3 + x_1). \tag{A2}$$

Insofar as t_1, t_2, t_3 are assumed to be different, one can show that there exists a function $F(t_1x_3 + x_1)$ (see below) such that (A2) is equivalent to the following equation:

$$(\partial_3 - t_2\partial_1)(\partial_3 - t_3\partial_1)(f - F(t_1x_3 + x_1)) = 0 \tag{A3}$$

where $(\partial_3 - t_2\partial_1)(\partial_3 - t_3\partial_1)F(t_1x_3 + x_1) = f_0(t_1x_3 + x_1)$.

Similarly, from (A3) it follows that there exists a function $G(t_2x_3 + x_1)$ such that

$$(\partial_3 - t_3\partial_1)(f - F(t_1x_3 + x_1) - G(t_2x_3 + x_1)) = 0.$$

Therefore, there is $H(t_3x_3 + x_1)$, such that

$$f - F(t_1x_3 + x_1) - G(t_2x_3 + x_1) - H(t_3x_3 + x_1) = 0. \qquad \square$$

Note that in the case when $t_2 = \bar{t}_2$ for f to be real it is necessary that $G = \bar{H}$, i.e. $f = F + 2\,\mathrm{Re}\,G$.

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